Projective Space *

1 Notations

$\hookrightarrow$: injection $\sim$: equivalence

Def: Homotopy, topologically equivalent

Def: Homeophorphism, f has a continuous inverse

Field: a field is a set on which addition, subtraction, multiplication, and division are defined and behave as the corresponding operations on rational and real numbers do.

Ring: fields where multiplication need not be commutative and multiplicative inverses need not exist. A ring is a set equipped with two binary operations satisfying properties analogous to those of addition and multiplication of integers.

2 Differential Manifolds

Def: M is a dimension n *Topological manifold* if M is a Hausdorff space, second-countable, locally euclidean of dimension n.

Def: Hausdorff, for any p, q, we can find neighborhoods that do not intercept.

Def: second countable, countable basis for the topological space

Def: locally euclidean of dimension n, for each point $p \in M$, there exists a homeomorphism $\varphi : U \to \hat{U}$, where U is the nbhd of p, $\hat{U}$ is an open set in $\mathbb{R}^n$. coordinate chart, (u, φ)

Note: every topological manifold has only one dimension. There is no notion of smoothness for the topological manifold.

Differentiable Geometry: try to describe the smoothness over the manifold without referring to the ambient Euclidean space, so we define a smooth structure such that: any two coordinate charts (u, φ) and (v, ψ), we require $\varphi^{-1} \circ \psi$ is a smooth homeomorphic, i.e. diffeomorphism (bijection, smooth, smooth inverse)

Smoothness: by default C^∞, C^1 can be extended to C^∞ somehow.

Def: atlas of M to be a collection of charts whose domains cover M. Each chart maps to the same dimension of Euclidean space.

Def: an atlas is called smooth if any two charts in A are smoothly compatible

Def: A smooth structure on M is a maximal smooth atlas: differentiable structure

Note: there can be many smooth structures on a manifold. Two smooth atlases define the same smooth structure iff their union is a smooth atlas.

Eg: $\psi : \mathbb{R} \to \mathbb{R}$ by $x \to x$, $\varphi : x \to x^3$. is a smooth map whose inverses are smooth in the sense of their smooth structures.

Note: the same topological space can have distinct (”nondiffeomorphic”) smooth structures. Lowest dimension this happens is 4. Easiest dimension to see it is 7.

Note: big open problem: are there exotic smooth structure on $\mathbb{S}^4$?

Eg: 1. finite collection of points. 2. $\mathbb{R}^n$. 3. $\mathbb{R}$, $\varphi : x \to x^3$. 3. Finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}^1$. 4. spheres. 5. $\mathbb{RP}^n$ = all lines in $\mathbb{R}^{n+1}$, $\mathbb{CP}^n$ = all ”lines” in $\mathbb{C}^{n+1}$. 6. Grassmannian. 7. SO(n): rotation in n-dimensional euclidean space.

*Lee JM. Introduction to Smooth Manifolds. Changes. 2007 Mar 7.

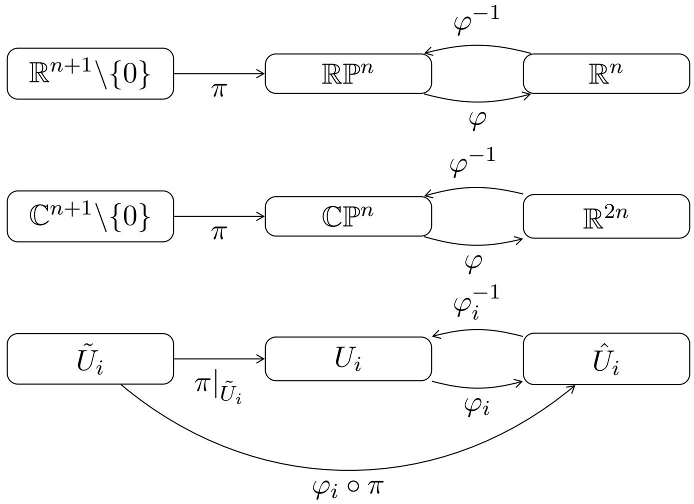

Figure 1: Construction of projective space

2.1 Projective Spaces

The projective space $\mathbb{RP}^n$ is a quotient space of $\mathbb{R}^{n+1}\backslash\{0\}$ by the equivalence:

$$(x^1, \ldots, x^{n+1}) \sim (\lambda x^1, \ldots, \lambda x^{n+1}), \lambda \in \mathbb{R}\backslash\{0\}.$$

Consider $\tilde{U}_i = \{(x^1, \ldots, x^{n+1}), x^i \neq 0\} \subset \mathbb{R}^{n+1}\backslash\{0\}$, and a quotient map $\pi_{\tilde{U}_i} : \tilde{U}_i \to U_i$,

$$\pi|_{\tilde{U}_i}(x^1, \ldots, x^{n+1}) = (x^1/x^i, \ldots, x^{i-1}/x^i, 1, x^{i+1}/x^i, \ldots, x^{n+1}/x^i),$$

and we let $\varphi_i : U_i \to \mathbb{R}^n$ be

$$\varphi_i(x^1, \ldots, x^{n+1}) \equiv \varphi_i(x^1/x^i, \ldots, 1, \ldots, x^{n+1}/x^i) = (x^1/x^i, \ldots, x^{i-1}/x^i, x^{i+1}/x^i, \ldots, x^{n+1}/x^i),$$

$$\varphi_i^{-1}(\hat{u}^1, \ldots, \hat{u}^n) = (\hat{u}^i, \ldots, \hat{u}^{i-1}, 1, \hat{u}^i, \ldots, \hat{u}^n).$$

Similar construction can be done for $\mathbb{CP}^n$ by writing elements in $\mathbb{C}^{n+1}$ as $(x^1, y^1, \ldots, x^{n+1}, y^{n+1})$. Define $\tilde{U}^i \subset \mathbb{C}^{n+1}$ such that $Z^i = x^i + iy^i \neq 0$. Let $\varphi_i : \mathbb{CP}^n \to \mathbb{R}^{2n}$ be

$$\varphi_i(Z^1, \ldots, Z^{n+1}) = (Z^1/Z^i, \ldots, Z^{n+1}/Z^i)$$

$$= (\frac{x^1 x^i + y^1 y^i}{\|Z_i\|_2}, \frac{x^i y^1 - x^1 y^i}{\|Z_i\|_2}, \ldots, 1, 0, \ldots, \frac{x^{n+1} x^i + y^{n+1} y^i}{\|Z_i\|_2}, \frac{x^i y^{n+1} - x^{n+1} y^i}{\|Z_i\|_2}),$$

and the inverse φ_i^{-1} is constructed by removing the two entries corresponding to Z_i.

To prove projective space is a smooth manifold, one must prove it is a topological manifold with a smooth structure. To prove it has a smooth structure, we only need to check if the transition map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is diffeomorphic.

$$\varphi_j \circ \varphi_i^{-1}(\hat{u}^1, \ldots, \hat{u}^n) = \varphi_j(\hat{u}^1, \ldots, \hat{u}^{i-1}, 1, \hat{u}^i, \ldots, \hat{u}^n)$$

$$= (\hat{u}^1/\hat{u}_j, \ldots, \hat{u}^{i-1}/\hat{u}_j, 1/\hat{u}_j, \hat{u}^i/\hat{u}_j, \ldots, \hat{u}^{j-1}/\hat{u}_j, \hat{u}^{j+1}/\hat{u}_j, \hat{u}^n/\hat{u}_j).$$